Divining Dividends

Alain Ouzou and Pedro Ferreira of ITO33 discuss the potential that lies in approaches to valuation of dividend protection clauses for convertible bonds

A convertible bond holder is short dividend. When a dividend is paid, the underlying share drops by a fraction of this dividend. Unlike the shareholder, the convertible bond holder does not receive this dividend. The spot moves away from the conversion price, making the probability that the bond will be in-the-money smaller. For the pricing of a convertible bond, the expectation of future dividends is a key parameter, which can have a big impact.

The reason behind dividend protection clauses

The dividend derivatives market has emerged over the last ten years and dividend futures and swaps are starting to trade with high liquidity and small bid/ask spreads for the major indices. OTC dividend swaps on single names are now available and are becoming increasingly popular. Dividends seem to have become a new genuine asset class. Nevertheless, many studies show that the dividend derivatives market can be mispriced, essentially due to structural bank positions, selling long-term maturities to hedge the dividend risk arising from sales of structured products.

Dividend protection features were introduced after a change in US dividend taxation rules,[1] increasing the benefits of dividend payments and resulting in higher dividends on the underlying shares. Convertible bond investors became increasingly aware of dividend risk, making the dividend protection clause the overall standard for convertible bond new issues.

Scope, literature survey, and objectives

The modelization of the dividend into the pricing engine has already been described in various books and academic papers. Nevertheless, even the most recently published book on convertible bonds (de Spiegeleer and Schoutens, 2011) devotes only three pages to the impact of the dividend on the pricing (pp. 194–196). In a recent paper about discrete dividends, Gocsei and Sahel (2010) provide a sound theoretical approach to the influence of dividends on the equity dynamics, but they explain in a note that their modelization "becomes less true when considering large maturities and dividends," which is mostly the case for convertible bonds. In this article, we will show explicitly that the way dividends are handled by the pricer strongly modifies the diffusion of the spot and therefore modifies the valuation of equity derivatives and especially convertible bonds, which are more sensitive to dividend risk due to their long maturity.

For technical articles on dividend protection, the literature is reduced drastically to almost nothing. The only academic paper (Mo, 2006) misses the key fact that the conversion ratio adjustment upon dividend is not determined by the initial spot at pricing date, but by the future (not known and therefore stochastic) underlying share price at the ex-dividend date: the problem becomes path-dependent. In this article, we will analyze explicitly the modelization of dividend protection clauses for convertible bonds. After a description of the different dividend protection features found in the prospectuses, we will emphasize how the dividend protection works and how it is possible to handle it.

Dividend protection types

Dividend protection is very widespread nowadays, nevertheless there is no standard lore on the vocabulary and description of features in the prospectuses. We can distinguish two main protection types:

- **Conversion ratio adjustment.** The convertible bond holder will be compensated (in full or in part) by an increase in the number of underlying shares he will get when converting.
- **Dividend pass through.** The convertible bond holder will be compensated (in full or in part) by a cash amount proportional to the dividend multiplied by the fixed conversion ratio.

The conversion ratio adjustment clause is currently the most common. Only a few old convertible bond issues have dividend pass through protections, therefore we will focus only on the conversion ratio adjustment clause and analyze it in detail.

Conversion adjustment

The conversion ratio adjustment is implemented by multiplying the conversion ratio prevailing just before the dividend by a fraction designed to compensate globally the effect of the dividend. There are several formulae which are not rigorously equivalent but which do not differ significantly for small dividends. A full protection formula should let the parity remain unchanged on both sides of the dividend payment. We can distinguish the following four formulae:

Standard:

$$\mathrm{CR}_{\mathrm{NEW}} = \mathrm{CR}_{\mathrm{OLD}} \frac{S_0}{S_0 - E} \tag{1}$$

Modified:

$$\mathrm{CR}_{\mathrm{NEW}} = \mathrm{CR}_{\mathrm{OLD}} \frac{S_0 + E}{S_0} \tag{2}$$

Conversion price:

$$\mathrm{CR}_{\mathrm{NEW}} = \mathrm{CR}_{\mathrm{OLD}} \frac{\mathrm{CP}_{\mathrm{OLD}}}{\mathrm{CP}_{\mathrm{OLD}} - E} \tag{3}$$

and Ex-dividend:

$$\mathrm{CR}_{\mathrm{NEW}} = \mathrm{CR}_{\mathrm{OLD}} \frac{S_0 - D + E}{S_0 - D}, \tag{4}$$

along with the following notations:

- $\mathrm{CR}_{\mathrm{OLD}}$: the conversion atio prevailing before the ex-dividend date.
- $\mathrm{CR}_{\mathrm{NEW}}$: the conversion ratio prevailing after the ex-dividend date (including the adjustment).
- S_0: the spot prevailing before the ex-dividend date.
- D: the dividend.
- E: the part of the dividend that is protected (equal to D for full protection).
- $\mathrm{CP}_{\mathrm{OLD}}$: the conversion price prevailing before the ex-dividend date.

Note that the four formulae can be written differently in the prospectuses:

- The formulae can refer to the conversion price instead of the conversion ratio. In this case the fraction is simply inverted, since $\mathrm{CP} = \mathrm{nominal}/\mathrm{CR}$.
- The formulae can refer to the cash dividend (expressed as an amount) or to the dividend yield (expressed as a percentage of spot). The difference between cash dividend and dividend yield raises several methodological questions about modelization and will be discussed in more detail later in this article.

Pass through

The dividend pass through protection stipulates to pass the dividend in full or in part to the bond holder as an additional cash distribution. This additional cash distribution can be paid at the ex-dividend date or at the next coupon date. The formula is given by:

$$\text{additional cash} = c\,\mathrm{CR}\,D \tag{5}$$

With the following notation:

- c: the fraction of the dividend that is protected (equal to 1 for full protection).
- CR: the prevailing conversion ratio.
- D: the dividend.

Complex dividend protection issues

Dividend protection clauses include complex features that govern the activation of the protection mechanism with the formulae detailed above. The prospectuses stipulate several conditions that have to be met to perform the adjustment. The vocabulary is complex and not unified, but we can distinguish three main condition types:

Barriers and triggers

The activation of the protection mechanism may be conditional on the dividend exceeding a certain level. This minimum dividend level that activates the protection is called dividend trigger. The prospectuses often refer to a unique dividend trigger, but they can also stipulate a schedule of dividend triggers anticipating a dividend growth.

There can be administrative limits on the minimum adjustment of the conversion ratio that may be performed. When the adjustment does not lead to a change of the conversion ratio higher than a specified threshold (generally 1%), the protection mechanism is postponed. The conversion ratio can be limited to a cap above which no further adjustment will be made (generally x% above the initial conversion ratio).

All these conditions act like knock-in and knock-out barriers on the dividend and the conversion ratio.

Annual triggers

The activation of the protection is not conditional on each dividend but depends on the sum of the dividends over a period, typically over the fiscal year. If the current sum of the dividends since the beginning of the lookup period is higher than a value, called the annual trigger, the convertible bond holder is compensated by the application of the protection mechanism.

Carry forward

When all activation conditions are not met, the adjustment is not performed and is postponed to a future date. Once all conditions are met to activate the protection, all past unperformed adjustments are added and applied according to the formulae. Such adjustments are said to be carry forward.

Cross currency

Cross-currency convertible bonds are bonds denominated in a first currency that can be converted into underlying shares denominated in a second currency. The underlying share itself can induce an additional cross-currency effect: the dividend can be denominated in a third currency. Cross-currency aspects have to be taken into account carefully.

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Market practices for modeling future dividends

With increasing liquidity of dividend swaps and dividend futures, dividends seem to have become an obvious parameter of the equity derivatives pricer. The next dividend is already announced and traders usually extrapolate the following dividends with a growth rate or use directly the dividend expected by the market of dividend derivatives.

However, at the level of the pricing model, continuous, proportional, or discrete dividends are not equivalent and modify the spot diffusion process which leads to different implied volatility, greeks, and hedge ratios.

Continuous dividends (repo)

The simplest modelization of future dividends consists of supposing a continuous payment, often described by a term structure. This approach is not appropriate for handling dividend protection clauses, because the real dividend protection adjustment is applied at discrete times and not continuously.

Proportional dividends

The modelization of dividends with proportional dividends consists of assuming that the share price follows a lognormal diffusion between the ex-dividend dates and jumps by an amount proportional to the share price at the ex-dividend date. This modelization reflects the discontinuity of the share price at the ex-dividend date and is therefore compatible with a dividend protection framework. In the market, the correlation between the dividend amount and the stock price can be significant, therefore assuming a constant dividend yield (or a term structure of dividend yields) may seem a good estimation of the future dividend schedule.

Because the size of the jump is proportional to the share price, the overall spot distribution is not strongly modified. Implementation of the proportional dividend (also known as discrete dividend yield) has been studied in many research papers and will not be analyzed in detail in this article. Only the dividend protection mechanism will be studied in the following sections.

Cash dividends

In this section we will not talk about an *n*th approximation of the Black–Scholes toy model with esoteric closed formulae, but will explore how discrete cash dividends modify the spot diffusion used inside the pricing engine and analyze the consequences for implied volatility, greeks, and hedge ratios.

Firstly, we will explain the impact of discrete cash dividends on the expected spot distribution. Unlike proportional dividends, cash dividends create a non-lognormal dynamic of the underlying. In particular, the probability that the stock price drops below the assumed cash dividend is not always zero and depends on the dividend level, spot volatility, and maturity of the derivative.

Secondly, we will explain in detail the influence of discrete cash dividends on the pricing of the equity derivative with emphasis on the implied volatility, the deformation of the greeks (especially delta), and the consequences for hedging purposes.

Unlike proportional dividends, cash dividends create a non-lognormal dynamic of the underlying. In particular, the probability that the stock price drops below the assumed cash dividend is not always zero and depends on the dividend level, spot volatility and maturity of the derivative

Spot distribution with cash dividends

The most realistic way to model discrete dividends is to assume that the stock price drops by a fraction[2] of the dividend at the ex-dividend dates and follows a usual lognormal diffusion process between ex-dividend dates.

Many research papers have tried to build other models which are not financially consistent and can lead to serious mispricing. The main justification of these models is essentially to stay in the Black–Scholes closed-form world without the need to resort to efficient numerical methods. Today, robust and fast numerical methods with PDE are available to market practitioners and accuracy should be the main concern.

Figure 1 shows a graphical illustration of the share price distribution for different dividend assumptions.

Probability that spot drops to zero

Figure 2 shows a graphical illustration of the probability that the share price drops to zero due to cash dividends. The surface shows the dependency of this probability with respect to the volatility and the maturity.

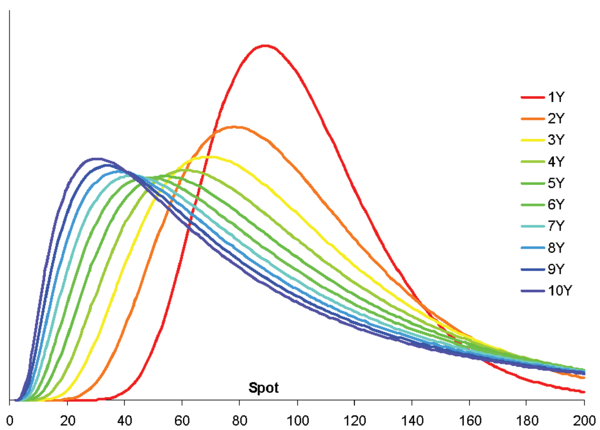
Defining the dividend policy

As seen before, for long maturities or when the volatility is high, the probability that the spot diffuses below the level of the cash dividend is not negligible. The dividend policy must then be clarified in order to ensure that the spot at least remains positive or that the share does not pay too high a dividend in relation to its price.

The most simple dividend policy ensuring a positive share price is

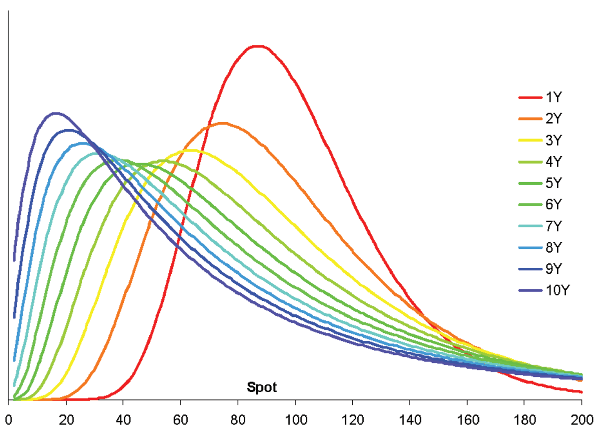
Figure 1: Spot distribution with cash dividend for several maturities

No dividend



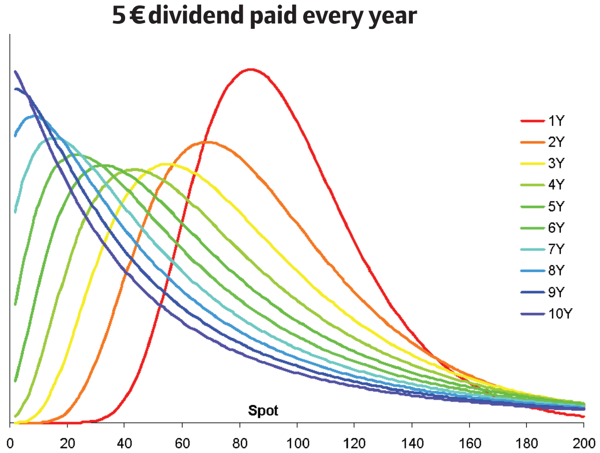
Without dividend, spot distribution follows the standard lognormal diffusion.

2€ dividend paid every year



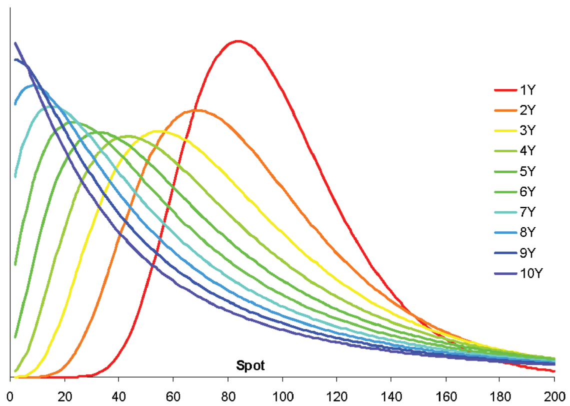
With 2€ dividend paid every year, the spot distribution is shifted by 2€ every year and then diffuses with a 30% volatility. The spot distribution looks globally lognormal, nevertheless when maturities are long the probability of very small spot values increases significantly.

5 € dividend paid every year



With 5€ dividend paid every year, the spot distribution is shifted by 5€ every year and then diffuses with a 30% volatility. For maturities after 3 years the distribution does not look lognormal and the probability of very small spot values increases drastically.

9€ dividend paid every year



With 9€ dividend paid every year, the spot distribution is shifted by 9€ every year and then diffuses with a 30% volatility. Even for short maturities, the distribution does not look lognormal and the probability of very small spot values increases significantly. Moreover, an important part of the distribution disappears while the maturity increases, which corresponds to the case where the stock price is below the next dividend: the stock price drops to zero and will not evolve anymore.

The spot distribution is plotted every year just after the ex-dividend date, for a maturity between 1 and 10 years.
Pricing parameters: r = 0%, initial spot = 100€, volatility = 30%.

known as the liquidator policy. The cash dividend D is paid until the spot becomes smaller than D, in this case the dividend becomes $d = S$ for $S < D$. The stock price is absorbed at zero:

$$d(S) = \begin{cases} D & \text{if } S \geq D \\ S & \text{if } S < D \end{cases} \tag{6}$$

Note that this payoff is equivalent to a deeply out-of-the-money (OTM) put with strike D.

Put–call parity revisited

The put–call parity without dividend states that:

$$\text{Call} - \text{Put} = S - \text{PV}(K) \tag{7}$$

where PV(K) is the present value of K, simply expressed as K discounted by the risk-free rate at the maturity of the options.

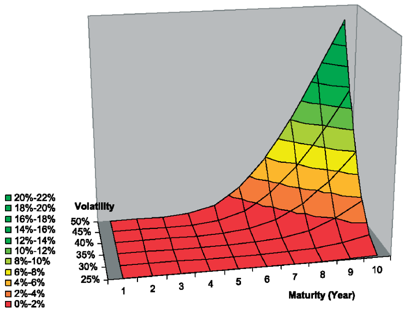
The cash dividend creates an issue. One is tempted to say that if one receives a fixed amount D at time t, the value of the dividend today should be: PV(D) = D B(t). But this simple analysis misses the fact that we cannot expect the dividend D to remain constant in case the stock price $S(t)$ falls close or below D at t, since the ex-dividend price would then be negative. As seen before, the dividend policy ensures a consistent share price is equivalent to a deeply OTM put option. In this case the call–put parity can be written:

$$\text{Call} - \text{Put} = S - K\,B(T) - (D\,B(t) - \text{Put}(D)) \tag{8}$$

With several dividends paid until maturity T, the put–call parity gets an additional term representing a strip of deeply OTM put options with strike equal to the cash dividend level D_i and maturity equal to the ex-dividend date t_i:

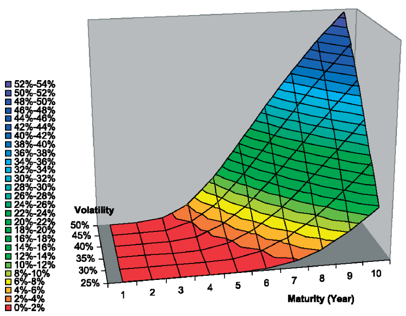
Figure 2: Probability that spot drops to zero as a function of maturity and volatility for different cash dividend assumptions.

2 € dividend paid every year



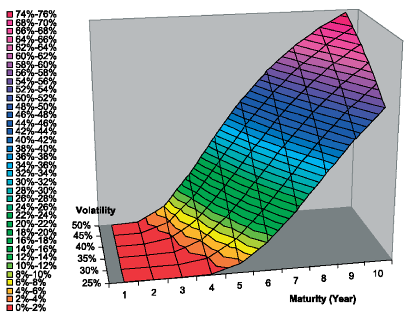
With 2€ dividend paid every year, the spot never drops to zero for typical listed options (maturity shorter than 5 years) and all volatility levels. Only long-maturity OTC options of highly volatile stocks are subject to a spot dropping to zero.

5 € dividend paid every year



With 5€ dividend paid every year, only maturities shorter than 3 years are preserved from the spot dropping to zero. For medium to long maturities, the probability that the stock drops to zero is absolutely not negligible and increases strongly with respect to the volatility.

9 € dividend paid every year



With 9€ dividend paid every year, only maturities shorter than 2 years are preserved from the spot dropping to zero. All other configurations of maturity and volatility lead to an important probability that the spot drops to zero.

The probability that the spot drops to zero is plotted every year just after the ex-dividend date, for a maturity between 1 and 10 years and a volatility between 25% and 50%.

Pricing parameters: r = 0%, initial spot = 100€.

$$\mathrm{Call}(K,T) - \mathrm{Put}(K,T) = S - K\,\mathrm{B}(T) - \sum_{\substack{i=1\\ t_n<T}}^{n} \left(D_i \mathrm{B}(t_i) - \mathrm{Put}(D_i, t_i)\right) \quad (9)$$

We identify two new issues in the put–call parity relationship:

- The put–call parity depends on the chosen dividend policy.
- The put–call parity depends on the value of a deeply OTM put.

Dividend schedule best practices

We strongly recommend using a cash dividend for short-maturity dividends (the announced dividend and possibly the next dividend forecast) and using a proportional dividend for long-maturity dividends.

The conclusion presented here is absolutely not revolutionary, because it is the tradional way traders model the dividend forecast. Nevertheless, we have proved in this section that this usual way is the only consistent method to take into account discrete deterministic dividends. This type of dividend schedule (cash and proportional) is correctly supported in all equity derivatives solvers. However, when dealing with dividend protection features, a cash and proportional dividend schedule will have special implications.

Handling of dividend protection clauses for convertible bonds

Tweaking dividend forecast

The simplest way to take into account the dividend protection is to tweak the dividend forecast used for the pricing. If the expected dividend is above the trigger, then the dividend forecast used by the convertible bond solver is capped to the trigger, simulating the protection. With this approach, the dividend protection is not a pricing feature that modifies the convertible pay-off but a simple matter of data entry.

Below the trigger, the convertible is not protected. Above the trigger, the convertible is not sensitive to the dividend. The validity of this modeling will be discussed here.

The dividend forecast used for pricing requires homogeneity with the trigger of the dividend protection. It is not possible to determine rigorously a proportional dividend yield for pricing when dividend protection is expressed as an absolute value.

Modeling the real protection mechanism

Adapting the numerical schema

The dividend protection mechanism for convertible bonds introduces some modifications of the partial differential equation (PDE) and the boundary conditions modeling the convertible bond. These changes impact the numerical schema used to solve the PDE.

In the simplest cases, new terms depending on the time and stock level are introduced; for example, the addition of a payment at dividend date or coupon date generated by the dividend pass through.

The protection by conversion ratio adjustment makes things more complex to handle. We observe that for the simplest cases the conversion ratio becomes a deterministic function of time and spot level parametrized by the dividend forecast. In the simple example where the bond is fully dividend protected, the formula (1) is used and the dividend forecast is given as only proportional dividends. In that case, for all dates after each dividend payment the conversion ratio becomes a simple function of time of the form $\mathrm{CR}_i = \mathrm{CR}_{i-1} \times \frac{1}{1-d_i}$, where d_i is the dividend at time t_i.

There are nevertheless cases where the dividend protection clauses introduce a dependency of the conversion ratio on the history of the share price, not on the spot at each grid point. This can be seen in the same simple example using the standard formula when the dividend forecast contains an absolute (cash) dividend. In that case the conversion ratio at time t and spot S depends on the stock level at the time of the previous dividend payment t_i, as a stock price of S at time t can be reached from different spot levels at time t_i. This requires the introduction of extra dimensions in the numerical schema in order to take into account all the possible paths for the stock level after each dividend payment.

Cash dividend protection

In this section, we calculate the new parity of a dividend protected convertible bond with cash dividend protection and dividend in cash on the underlying. We assume that the underlying pays one dividend per year. We consider the standard adjustment formula, which is the most frequent case in the convertible universe. Other formulae would provide different results, but the same behavior with respect to the spot, dividend, and trigger.

The following notation will be used:

- D: the dividend in cash.
- Tr: the trigger of the dividend protection, expressed as a cash amount.
- n: the number of dividends until maturity, equal to the maturity expressed in years (one dividend per year).
- S: the current spot level.
- S_i: the spot level prevailing at year i.
- CR_i: the conversion ratio prevailing at year i.

Applying the conversion ratio adjustment each year:

Year 1. Consider the payment of 1 dividend in 1 year. The new ratio is given by the adjustment formula and the new spot is given by the current spot minus the dividend:

$$\begin{aligned}&\mathrm{Parity}_{\mathrm{NEW}}(S, D, \mathrm{Tr}, 1)\\ &\quad= (S-D)\mathrm{CR}\frac{S}{S-(D-\mathrm{Tr})}\end{aligned} \tag{10}$$

Year 2. The spot prevailing before the second dividend is $S_1 = (S - D)$ and will become $S_2 = (S - 2D)$ after the dividend. The conversion ratio prevailing before the second dividend is

$$\mathrm{CR}_1 = \mathrm{CR}\frac{S}{S-(D-\mathrm{Tr})}$$

and will be multiplied by

$$\frac{S_1}{S_1-(D-\mathrm{Tr})} = \frac{(S-D)}{(S-D)-(D-\mathrm{Tr})}.$$

Finally, after the second dividend we obtain:

$$\begin{aligned}&\mathrm{Parity}_{\mathrm{NEW}}(S, D, \mathrm{Tr}, 2)\\ &\quad= S_2\mathrm{CR}_1\frac{S_1}{S_1-(D-\mathrm{Tr})}\end{aligned} \tag{11}$$

$$\begin{aligned}&= (S-2D)\mathrm{CR}\frac{S}{S-(D-\mathrm{Tr})}\\ &\quad\times\frac{S-D}{(S-D)-(D-\mathrm{Tr})}\end{aligned} \tag{12}$$

The dependency with respect to S, D, and Tr is absolutely not straightforward even for a 2-year maturity convertible bond with one dividend per year.

Year *n*. The generalization to year n is obtained by applying n times the adjustement formula for the conversion ratio and by taking into account the drop in spot due to the n dividends:

$$\begin{aligned}&\mathrm{Parity}_{\mathrm{NEW}}(S, D, \mathrm{Tr}, n)\\ &\quad= \frac{(S-nD)\mathrm{CR}}{\prod_{i=1}^{n}\left(1-\frac{D-\mathrm{Tr}}{S-(i-1)D}\right)}\end{aligned} \tag{13}$$

By performing a Taylor expansion of this function with respect to D, we obtain:

$$\begin{aligned}&\mathrm{Parity}_{\mathrm{NEW}}(S, D, \mathrm{Tr}, n)\\ &\quad= S\mathrm{CR}\left(K_0 + K_0K_1\frac{D}{S} + K_0K_1K_2\frac{D^2}{S^2}\right)\\ &\quad+o(D^2)\end{aligned} \tag{14}$$

with:

$$K_0 = \frac{1}{(1+(\mathrm{Tr}/S))^n} \tag{15}$$

$$K_1 = -\frac{n(n+1)}{2}\frac{\mathrm{Tr}/S}{1+\mathrm{Tr}/S} \tag{16}$$

$$\begin{aligned}K_2 = +\frac{2n+1}{3} - \frac{(n+2)(3n+1)}{12}\\ \times\frac{\mathrm{Tr}/S}{1+\mathrm{Tr}/S}\end{aligned} \tag{17}$$

Yield dividend protection

We can apply the same method for the dividend protection with trigger expressed in yield and proportional dividend expressed in yield.

We use the following notation:

- d: the proportional dividend in yield.
- tr: the trigger of the dividend protection, expressed as a yield amount.
- n: the number of dividends until maturity, equal to the maturity expressed in years (one dividend per year).
- S: the current spot level.

Parity at year k is obtained from parity at year $k-1$ by:

$$\begin{aligned}&\mathrm{Parity}_{\mathrm{NEW}}(S, d, \mathrm{tr}, k)\\ &\quad= S_k\mathrm{CR}_{k-1}\frac{S_{k-1}}{S_{k-1}-S_{k-1}(d-\mathrm{tr})}\end{aligned} \tag{18}$$

$$= S_{k-1}(1-d)\mathrm{CR}_{k-1}\frac{1}{1-(d-\mathrm{tr})} \tag{19}$$

$$= S_{k-1}\mathrm{CR}_{k-1}\frac{1-d}{1-(d-\mathrm{tr})} \tag{20}$$

By induction, we obtain the following equation:

$$\begin{aligned}&\mathrm{Parity}_{\mathrm{NEW}}(S, d, \mathrm{tr}, n)\\ &\quad= S\mathrm{CR}\left(\frac{1-d}{1-(d-\mathrm{tr})}\right)^n\end{aligned} \tag{21}$$

By performing a Taylor expansion of this function with respect to d, we obtain:

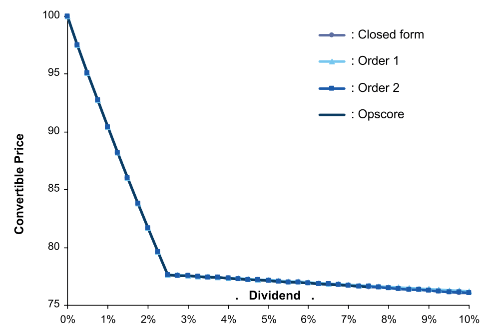
$$\begin{aligned}&\mathrm{Parity}_{\mathrm{NEW}}(S, d, \mathrm{tr}, n)\\ &\quad= S\mathrm{CR}\frac{1}{(1+\mathrm{tr})^n}\left(1-\frac{n\mathrm{tr}}{1+\mathrm{tr}}d\right)+o(d)\end{aligned} \tag{22}$$

$$\begin{aligned}&= S\mathrm{CR}\frac{1}{(1+\mathrm{tr})^n}\left(1-\frac{n\mathrm{tr}}{1+\mathrm{tr}}d\right.\\ &\qquad\left.-\frac{n\mathrm{tr}-\frac{n(n-1)\mathrm{tr}^2}{2}}{(1+\mathrm{tr})^2}d^2\right)\\ &\quad+o(d^2)\end{aligned} \tag{23}$$

Figure 3 shows parity as afunction of dividend.

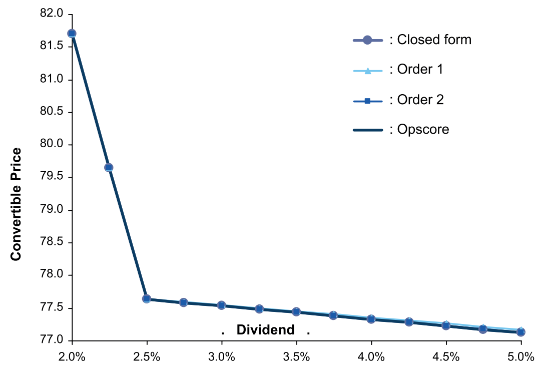
Figure 3: Parity as a function of dividend

Yield dividend protection with proportional dividend



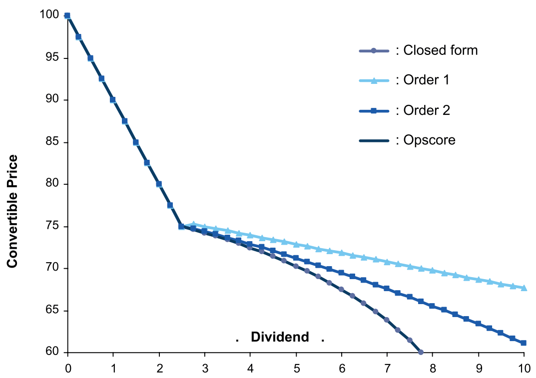
Parity as a function of the proportional dividend yield in the presence of yield dividend protection is perfectly similar even with the approximation at order one. Below the trigger (2.5%), the convertible is not protected and we observe the typical strong dependency of the price with respect to the dividend yield. Above the trigger, the convertible is protected and the price sensitivity to the dividend yield is very small. Nevertheless, one should note that the sensitivity to the dividend yield above the trigger is not zero.

Yield dividend protection with proportional dividend – zoom



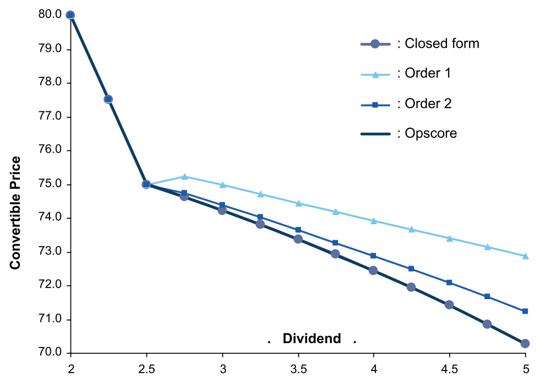
Above the trigger (2.5%) the increase in proportional dividend yield from 2.5% to 5% leads to a 0.51€ drop in price, while below the trigger the increase in dividend yield from 2% to 2.5% leads to a 4.07€ drop. The sensitivity of the parity with respect to the dividend yield is therefore 40 times smaller above the trigger: the convertible is protected.

Cash dividend protection with cash dividend



Parity as a function of the cash dividend level in the presence of cash (absolute) dividend protection is perfectly similar for Opscore and the closed formula. Taylor expansions at order one and order two provide only the order of magnitude of the remaining sensitivity, which is not negligible above the trigger.

Cash dividend protection with cash dividend – zoom



Above the trigger (2.5€) the increase in cash dividend from 2.5€ to 5€ leads to a 4.73€ drop in price, while below the trigger the increase in cash dividend from 2€ to 2.5€ leads to a 5.00€ drop. The sensitivity of the parity with respect to the cash dividend is therefore only 5 times smaller above the trigger: the convertible is not well protected.

Pricing parameters: maturity = 10 years, 1 dividend per year, r = 0%, initial spot = 100€, volatility = 1%, trigger = threshold = 2.5% or 2.5€

Comparison of the two approaches

In this section we compare the tweaked dividend forecast approximation to the rigorous full solving of the dividend protection problem. The previous part was dedicated to the change in parity, by paying special attention to the understanding of the protection mechanism through closed formulae. Here we will emphasize the complex behavior of the price obtained with the solver, by taking into account the role of the volatility.

Simple "homogeneous" cases

By "homogeneous" cases we mean cases where the protection trigger is of the same type (cash or proportional) as the dividend schedule.

Yield dividend protection with proportional dividend

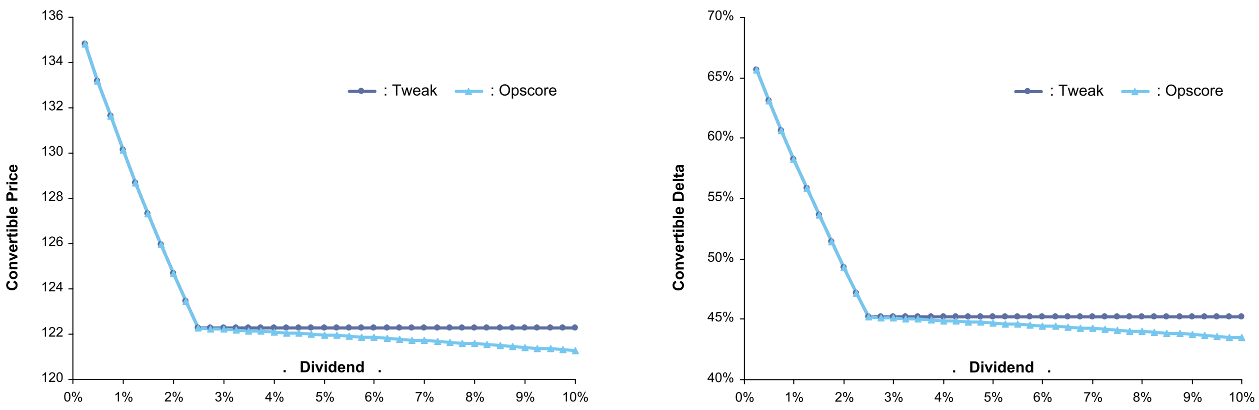
Yield dividend protection with proportional dividend is a real homogeneous problem (see Figure 4). It is possible to simplify the equations and to avoid the path dependency. This theoretical result allows us to evaluate rigorously the dividend protection feature by adjusting the dividend forecast with an adequate change of variable. This method is in fact used internally by the dividend approximation functionality in Opscore and provides perfect results. Here, the chosen simple adjustment method of the dividend forecast is to truncate the dividends exceeding the trigger.

Cash dividend protection with cash dividend

The dependency of the price with respect to the cash dividend level differs strongly between the full solving

Figure 4: Yield dividend protection with proportional dividend

Price as a function of dividend yield | **Delta as a function of dividend yield**



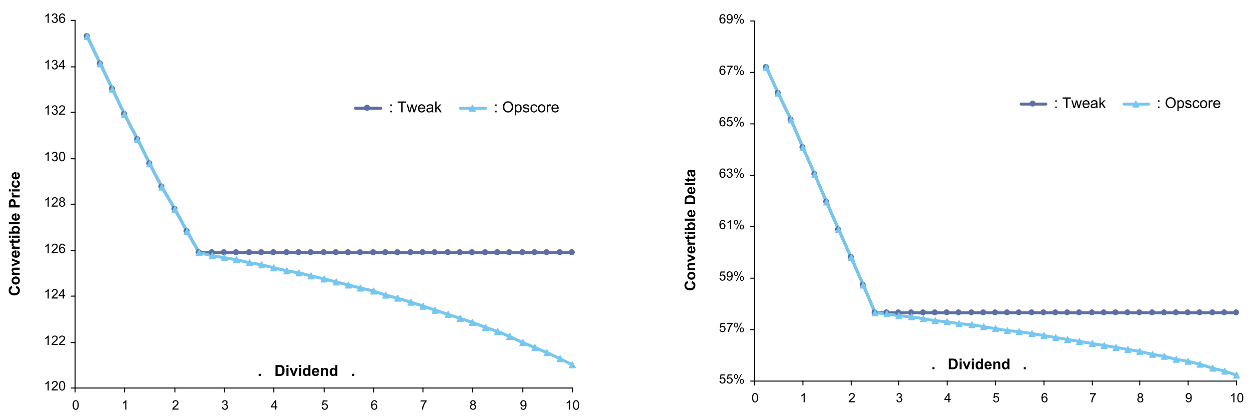
The price as a function of the proportional dividend yield in the presence of yield dividend protection is very similar for the full solving and for the approximation.

The delta as a function of the proportional dividend yield in the presence of yield dividend protection is very similar for the full solving and for the approximation.

Pricing parameters: maturity = 10 years, 1 dividend per year, r = 0%, initial spot = 100€, volatility = 30%, trigger = threshold = 2.5%

Figure 5: Cash dividend protection with cash dividend

Price as a function of cash dividend | **Delta as a function of cash dividend**



The price of the convertible bond in the presence of cash dividends and cash dividend protection shows an important remaining sensitivity for values above the trigger. This is due to the fact that the new parity is not properly compensated by the conversion ratio adjustment in the presence of a trigger and a threshold.

We observe that the disparities in price induce a difference in the convertible bond deltas.

Pricing parameters: maturity = 10 years, 1 dividend per year, r = 0%, initial spot = 100€, volatility = 30%, trigger = threshold = 2.5% or 2.5€

and the basic dividend tweaking method. In fact, above the trigger, the rigorous pricing of the cash dividend protection with cash dividend exhibits a non-negligible remaining sensitivity to the dividend, while the basic approximation leads to a constant price (see Figure 5). This sensitivity to the dividend is, however, attenuated by the volatility compared to the impact on the parity presented in the previous subsection.

It is strongly inconsistent to switch from a cash dividend model to a proportional dividend model because the trigger of the dividend protection is expressed as a cash amount or a yield. As described earlier, the type of dividend (proportional or cash) has a strong influence on the underlying share diffusion and therefore on the value of the embedded conversion option.

Practical cases

The dividend schedule used in this section is the typical parametrization: the next dividends are announced and can be considered as certain cash values, while the future mid-term and long-term dividends depend strongly on the future share prices but can be evaluated by a simple yield (or even a schedule of yields).

Yield dividend protection with cash and proportional dividend schedule

Because a proportional dividend with yield dividend protection is well approximated by the dividend tweaking method, it is expected that the first cash dividends do not modify the behavior of the price and greeks strongly. In fact, the diffusion of the spot price before the ex-dividend date does not lead to an important change of the equivalent dividend yield that matches the cash dividend.

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In Figure 6, the dividend level $x\%$ means the following schedule:

- x€ cash dividend for year 1.
- x€ cash dividend for year 2.
- $x\%$ proportional dividend yield for year n, $n > 2$.
- *We assume a spot level of 100€ to guarantee that the scale is relevant.*

Cash dividend protection with cash and proportional dividend schedule

Cash dividend protection with a mixed dividend schedule raises several questions (see Figures 7 and 8):

- Cash dividend with cash dividend protection is not a simple problem and the behavior of the price with respect to the cash dividend level above the trigger is not straightforward.
- Proportional dividends, which are the major part of the schedule for long-maturity convertible bonds, are not homogeneous with the trigger of the protection expressed in cash. When using a proportional dividend, the size of the jump in the underlying share ex-dividend date depends on the share price, which follows a diffusion law. The cash trigger of the protection behaves like a barrier, having an effect similar to a put option on the dividend level.

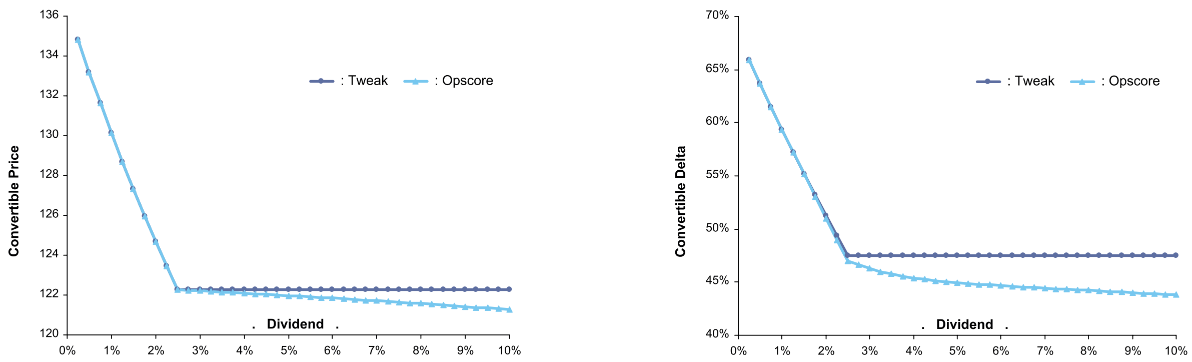
Real market cases

Why dividend protection features do not currently lead to serious mispricing

The average convertible delta is currently in Q2 2011 smaller than 50%, which means that most of the convertible bond universe is OTM or ATM. Due to the market turmoil in 2008, dividends have been cut and most of the current dividends

Figure 6: Cash dividend protection with dividend schedule (cash and proportional)

Price as a function of dividend level | **Delta as a function of dividend level**



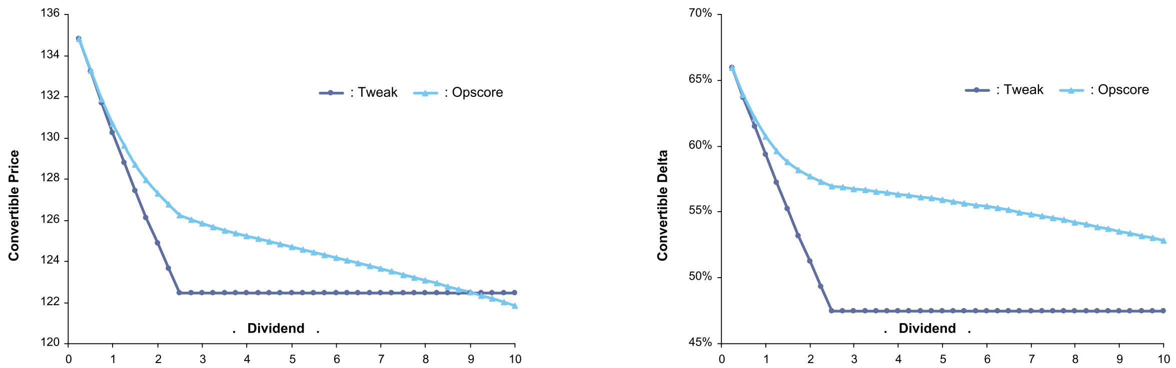
The price as a function of the dividend level in the presence of yield dividend protection is very similar between the full solving and the approximation. In fact, compared with the proportional dividend case, the dividend schedule differs only by the first two cash dividends.

Delta is correctly approximated by the tweaking dividend method but is always overestimated. The order of magnitude of the error is not relevant for outright investments but can be significant for delta hedged strategies, all the more so as the dividend exceeds the trigger.

Pricing parameters: maturity = 10 years, 1 dividend per year, r = 0%, initial spot = 100€, volatility = 30%, trigger = threshold = 2.5%

Figure 7. Cash dividend protection with dividend schedule (cash and proportional)

Price as a function of dividend level | **Delta as a function of dividend level**

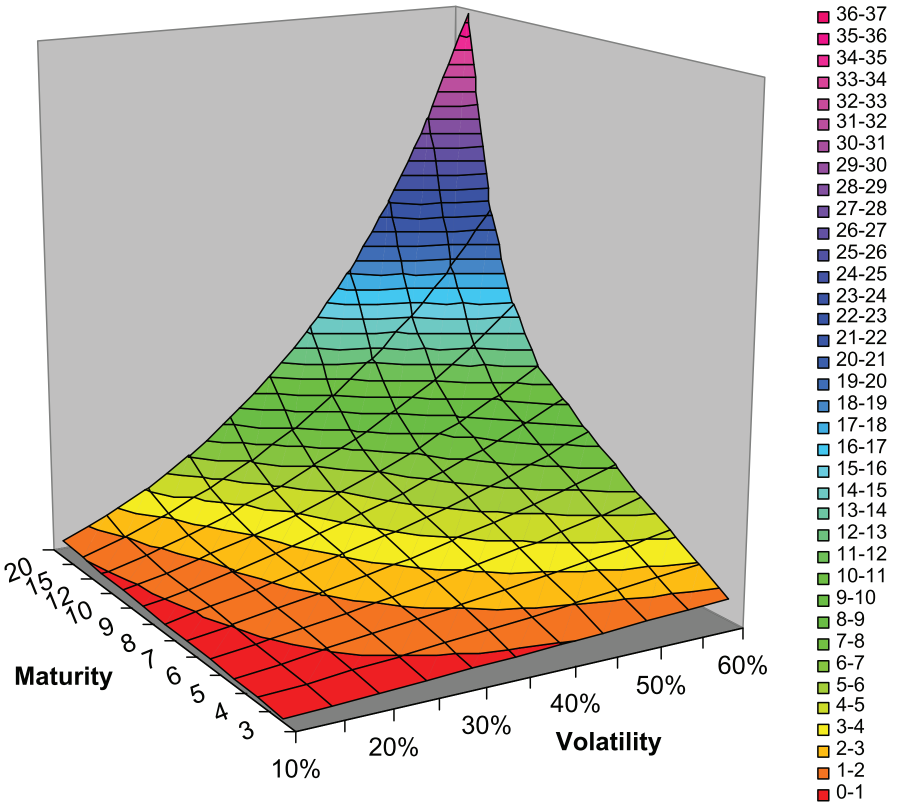


We see clearly that the convertible bond price as a function of the dividend level exhibits the typical convexity of a put option. This behavior cannot be properly approximated by the dividend tweaking method, which is far away from the rigorous method and leads to serious mispricing, especially when the dividend is close to the trigger.

The convertible bond price difference implies a very important discrepancy in the delta. The dividend tweaking method drastically underestimates the delta and cannot be used for hedging the sensitivity of the underlying.

Pricing parameters: maturity = 10 years, 1 dividend per year, r = 0%, initial spot = 100€, volatility = 30%, trigger = threshold = 2.5€

Figure 8: Difference between Opscore and Tweak



Description

The plot shows the difference[a] between the price computed with the full solving of the dividend protection in Opscore and the price calculated by tweaking the dividend forecast. The difference depends on the level of the dividend, the moneyness, the maturity, and the volatility.

- We chose a convenient value for the dividend, equal to the trigger, which is often the case at issuance.
- For the moneyness, we chose an initial spot (equal to parity) of 100, which provides a good proxy of an ATM (balanced) convertible with a fixed simple reference. OTM convertible bonds will be less sensitive to the dividend, while ITM convertible bonds will be more sensitive.

Results

- For maturities shorter than 5 years, the difference between the full solving and the dividend tweaking method is less than 3 volatility points, for an ATM convertible bond with a typical volatility between 30% and 40%. This important result explains why the dividend tweaking method is commonly used by market practitioners without exhibiting serious mispricing.
- For long-maturity convertible bonds and/or highly volatile stocks, the difference can be significant. Nevertheless, in these market configurations, the convertible bond is also sensitive to other factors such as the expected dividend level or the credit spread. The mispricing is therefore not easy to detect and can be hidden by other pricing parameters.

Pricing parameters: 1 dividend per year, r = 0%, initial spot = 100€, trigger = threshold = 2.5€, Dividend schedule: 1Y: 2.5€, 2Y: 2.5€, nY: 2.5%

[a]Difference is expressed in volatility points.

are smaller than the dividend at the time of issuance, which is often taken as reference for the trigger and threshold of the dividend protection. With current low dividend forecasts, we observe low dividend protection effects on convertible bond prices.

Why rigorous pricing of dividend protection will be essential if the market moves up

If the market moves up or anticipates a substantial increase in dividend levels, several factors will have a simultaneous influence on the pricing of the dividend protection:

- Convertible bonds will be more sensitive to the underlying (delta will be higher) and therefore will be more sensitive to the dividend (mu is higher).
- Vega will become smaller as the convertible bonds become more equity-like, causing the implied volatility to differ strongly between the full solving and the approximation of the dividend protection.
- The significant correlation between share price and dividend level will lead to an increase in the dividend, substantially exceeding the trigger.

Conclusion

This study shows that for complex long-maturity instruments such as convertible bonds, the modelization of the dividend forecast has a major impact on the valuation.

The presence of cash dividends in the forecasts introduces a significant deformation of the dynamics of the underlying, as shown in Figure 1. Long-date cash dividend forecasts should therefore be avoided.

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Dividend protection clauses do not fully remove the dividend sensitivity of the convertible bond, this is particularly true if cash dividend forecasts are used or if the dividend protection threshold is expressed as an absolute cash amount.

This study was performed using a model with deterministic dividend forecasts. Apart from the small effect of the dividend policy described in the paragraph on proportional dividends, the dividend will always be detached at the ex-dividend date with a perfectly known amount for a cash dividend schedule or with a perfectly known percentage of the future spot price for a proportional dividend schedule. With a proportional dividend, which should generally represent the major part of the schedule, the stochasticity is introduced: the dividend amount follows the same diffusion law as the stochastic process describing the share price. Due to the triggers and thresholds in the dividend protection clauses, this diffusion of the dividend amount on both sides of the barriers creates an additional convexity of the convertible price with respect to the dividend amount, as seen in the section on cash dividend protection with cash and proportional dividend schedule. This convexity is never shown by basic approximations, which lead to serious mispricing of the dividend protection feature.

When dealing with dividend convexity, one is tempted to talk about dividend volatility. With proportional dividends, dividend volatility is simply defined through the share volatility, which seems to be a coherent measure because of the strong observed correlation between dividend amounts and share prices. Nevertheless, the uncertainty in the future dividends and the difficulty of characterization of the dividend behavior suggests that using a model with full stochastic dividends may be interesting.

This can potentially open a new field for research: an ideal parametrization of the dividend process should take into account several regimes that reflect the dividend policy. If we say that dividends are becoming an asset class per se, we can then accept the idea that lower share prices need not necessarily be associated with lower dividends. Higher share prices may also be consistent with unchanged or even lower dividends.

This ideal model will therefore diffuse the several possible regimes and attribute a weight to them by a simple co-calibration on several market derivatives, and then will be able to evaluate the price of the contingent instrument.

Opscore is the complete front-office solution developed by ITO33 for the pricing, hedging, and analysis of convertible securities. It consists of three components: a data model of terms and conditions, a pricing engine, and an Excel front-end. More information about Opscore is available at http://www.ito33.com/opscore.

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ENDNOTES

1. Jobs and Growth Tax Relief Reconciliation Act, 2003.
2. Fraction is due to tax policies.

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